Advanced q-Bessel Function of Two Variables

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Abstract :- The main object of this paper to construct q-Bessel functions of two variables of first kind and found various results of $J_{r,s}$ (x, y ; q) like generating function, recurrence relation. Furthermore we use q-analogue to find some new significant results and generalizations have been discovered.

Keyword:- Bessel function of one variable, Bessel function of two variable, differential equation, Factorial Notation, generating function, recurrence relations, q-analogy.

I. INTRODUCTION

The problem of Mathematical Physics leads us to determine the solutions of Differential Equations which satisfy certain prescribed conditions. Many special functions have been exposed to current generalizations to a base of q, which are usually noted as q –special function. Basic analogise of Bessel functions have been introduced by Jackson [4] and Swarthow[9].

we know that [1]the ordinary cylindrical Bessel functions, define q-generalization of power series expansions. Three different types of such q-expansion can be recognized, each of them satisfy recurrence relations, second order q-differential equation and addition theorems, which reduce to those holding for the usual Bessel function in the limit q → 1.This paper presents new form of Bessel function with q –analogues of one and two variables.

II. PRELIMINARY NOTATIONS AND DEFINITIONS

Before entering the specific topic of the paper, let us briefly review the properties of q – Bessel functions. We discuss Definitions and Notations of q-analogy.

The q – shifted factorial Notation of Real and Complex number is given by [1].

(1)

$$(a;\ q)n = \begin{cases} 1 & n = 0 \\ \prod_{J=0}^{n-1}\{(1-aq^J)\} & \text{for } n\epsilon N \end{cases}$$

(2)

$$(a;\ q)\infty = \prod_{J=0}^{\infty}\{(1-aq^J)\}$$

$$(a;\ q)0 = 1$$

The q-factorial $[n]_{q\,!}$ being defined as, where n is integer

(3)

$$[n]_{q\,!} = \frac{(q;q)_n}{q^{\frac{n^2}{2}}}$$

Where 0 < q < 1.

The Bessel's function of first kind of order r is defined by [7]

(4)

$$J_r(x) = \sum_{m=0}^{\infty} \frac{m}{!\,m \quad \Gamma r+m+1}\left(\frac{x}{2}\right)^{2m+r}$$

The generalized q-Bessel function of one variable is defined by [1]

(5)

$$J_r(x;q) = \sum_{m=0}^{\infty} \frac{(-1)^m}{[m]_{q\,!}[\ r+m\ ]_{q}!}\left(\frac{x}{2}\right)^{2m+r}$$

When q → 1, (4)

And Two variable Bessel's functions are defined by [7] with r and s are integer we have the representations

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(6)

$$J_{r,s}(x,y) = \frac{\left(\frac{x}{2}\right)^r \left(\frac{y\,\rho(x)}{2}\right)^s}{\Gamma r+1 \quad \Gamma s+1}\, 0F_1\left(-; r+1; -\frac{x^2}{4}\right) 0F_1\left(-; s+1; -\frac{y^2\rho^2(x)}{4}\right)$$

or

$$J_{r,s}(x,y) = \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n}}{!n\ \ !m\ \ \Gamma r+m+1\ \ \Gamma s+n+1}\left(\frac{x}{2}\right)^{2m+r}\left(\frac{y\,\rho(x)}{2}\right)^{2n+s}$$

In this paper we have to prove that q-Bessel function of two variable generating functions, recurrence relation and some results of

$$J_{r,s}(x,y\,;q) = \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n}\left(\frac{x}{2}\right)^{2m+r}\left(\frac{y\rho(x)}{2}\right)^{2n+s}}{[m+r]_q!\ [m]_q!\,[n+s]_q!\ [n]_q!}$$

When $q \to 1$

III. GENERATING FUNCTION FOR $J_{r,s}(x\,,y\,;q)$

Definition:-For the advanced q- Bessel function of two variable $J_{r,s}(x\,,y\,;q)$, we defined by

(7)

$$J_{r,s}(x\,,y\,;q) = \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n}\left(\frac{x}{2}\right)^{2m+r}\left(\frac{y\rho(x)}{2}\right)^{2n+s}}{[m+r]_q!\ [m]_q!\,[n+s]_q!\ [n]_q!}$$

$$J_{r,s}(x\,,y\,;q) = \frac{\left(\frac{x}{2}\right)^r\left(\frac{y\rho(x)}{2}\right)^s}{[r]_q!\,[s]_q!}\ oF_1\left(-;[m]_q!;\frac{-x^2}{4}\right)oF_1\left(-;[n]_q!;\frac{-y^2\rho(x)^2}{4}\right)$$

Now we will deduce the generating function of the advanced q-Bessel function of two variables Jr,s(x,y;q).

Theorem 3.1 :- Prove that $J_{r,s}(x,y;q)$ is the coefficient of $t^r w^s$ in the expansion of

$$E_q\left[\left(\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y\rho(x)}{2}\left(w-\frac{1}{w}\right)\right)\right]$$

for $t \neq 0$, $w \neq 0$ and $t,w \in \mathbb{C}$.

Proof: - we know that by [2] there are two important special cases of the q- exponential function

(8)

$$E_q(x) = \sum_{r=0}^{\infty}\frac{x^r}{[r]_q!}$$

and

(9)

$$E_q(x) = \sum_{r=0}^{\infty}\frac{q^{r^2/2}\ \ x^r}{\langle q;q\rangle_r}$$

$|x| < 1$.

Consequently in the limit$q \to 1$, we have $\lim_{q\to 1}\left(E_q(1-q)z\right) = e^z$ by [2]

Using the formula (8), we get L.H.S.

$$E_q\left[\left(\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y\rho(x)}{2}\left(w-\frac{1}{w}\right)\right)\right]$$

$$= E_q\left[\left(\frac{xt}{2}\right)\right]E_q\left[\left(-\frac{x}{2t}\right)\right]E_q\left[\left(\frac{y\rho(x)w}{2}\right)\right]E_q\left[\left(-\frac{y\rho(x)}{2w}\right)\right]$$

$$= \sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(\frac{xt}{2}\right)^r\left(-\frac{x}{2t}\right)^m}{[r]_q!\ [m]_q!}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(\frac{y\rho(x)w}{2}\right)^s\left(-\frac{y\rho(x)}{2w}\right)^n}{[s]_q!\ [n]_q!}$$

If we replace r by m+ r & s by n + s, then we get following equation

$$= \sum_{r=-\infty}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{xt}{2}\right)^{m+r}\left(\frac{x}{2t}\right)^m}{[m+r]_q!\ [m]_q!}\sum_{s=-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{y\rho(x)w}{2}\right)^{n+s}\left(\frac{y\rho(x)}{2w}\right)^n}{[n+s]_q!\ [n]_q!}$$

$$= \sum_{r,s=-\infty}^{\infty}\sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}\left(\frac{x}{2}\right)^{m+r+m}\left(\frac{y\rho(x)}{2}\right)^{n+s+n}(w)^{n+s-n}\,(t)^{m+r-m}}{[m+r]_q!\ [m]_q!\,[n+s]_q!\ [n]_q!}$$

$$= \sum_{r,s=-\infty}^{\infty} t^r w^s \sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}\left(\frac{x}{2}\right)^{2m+r}\left(\frac{y\rho(x)}{2}\right)^{2n+s}}{[m+r]_q!\ [m]_q!\,[n+s]_q!\ [n]_q!}$$

(10)

$$E_q\left[\left(\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y\rho(x)}{2}\left(w-\frac{1}{w}\right)\right)\right]=\sum_{r,s=-\infty}^{\infty} t^r w^s\, J_{r,s}(x\,,y\,;q)$$

Which is a generating function of $J_{r,s}(x\,,y\,;q)$ .If using (9) then we get result.

(11)

$$E_q\left[\left(\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y\rho(x)}{2}\left(w-\frac{1}{w}\right)\right)\right] = q^{\frac{r^2+s^2}{2}}\sum_{r,s=-\infty}^{\infty} J_{r,s}(x\,,y : t\,,w\,;q)$$

Theorem3.2 The function $J_{r,s}(x\,,y\,;q)$ is q –analogy of each of the Bessel function an

$\lim_{q\to1} J_{r,s}\big((1-q)x\,,(1-q)y\,;q\big) = J_{r,s}\,(x,y).$

Proof: - Some Previous result of q-analogy by Exton [6] presented the q-exponential function, they say (12)

$$[n]_{q\,!}=\frac{(q;q)_n}{(1-q)^n}$$

$$[n+k]_{q\,!}=\frac{(q;q)_{n+k}}{(1-q)^{n+k}}$$

Where $0 < q < 1$.

Using the relation by Gasper [8]

(13)

$$(q;q)_{n+k} = (q;q)_k(q^{k+1};q)_n$$

$$\lim_{q\to1} J_{r,s}\big((1-q)x\,,(1-q)y\,;q\big)$$

$$=\lim_{q\to1}\sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}(1-q)^{m+r}(1-q)^{n}(1-q)^{m}\,(1-q)^{n+s}\,\left(\frac{x}{2}\right)^{2m+r}\left(\frac{y\rho(x)}{2}\right)^{2n+s}}{(q;q)_{m+r}\,\,(q;q)_m(q;q)_{n+s}(q;q)_n}$$

Using the relation (13), we obtain

$$=\lim_{q\to1}\sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}(1-q)^{2m}(1-q)^{2n}(1-q)^{r}\,(1-q)^{s}\,\left(\frac{x}{2}\right)^{2m+r}\left(\frac{y\rho(x)}{2}\right)^{2n+s}}{(q;q)_m(q;q)_r\,\,(q^{r+1};q)_m(q^{s+1};q)_n(q;q)_s\,(q;q)_n}$$

$$=\lim_{q\to1}\frac{(1-q)^r\,(1-q)^s}{(q;q)_r\,(q;q)_s}\sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}(1-q)^{2m}(1-q)^{2n}\left(\frac{x}{2}\right)^{2m+r}\left(\frac{y\rho(x)}{2}\right)^{2n+s}}{(q;q)_m\,(q^{r+1};q)_m(q^{s+1};q)_n\,(q;q)_n}$$

We know that by relation of q-gamma function with q-factorial function

(14)

$$\Gamma\,q(n+1)=[n]_{q!}$$

$$=\lim_{q\to1}\frac{1}{[r]_{q!}[s]_{q!}}\sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}(1-q)^{2m}(1-q)^{2n}\left(\frac{x}{2}\right)^{2m+r}\left(\frac{y\rho(x)}{2}\right)^{2n+s}}{(q;q)_m\,(q^{r+1};q)_m(q^{s+1};q)_n\,(q;q)_n}$$

Taking limit $q\to1$ and we obtain

$$=\frac{1}{[r]_![s]_!}\sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}\left(\frac{x}{2}\right)^{2m+r}\left(\frac{y\rho(x)}{2}\right)^{2n+s}}{(1)_m\,(r+1)_m(s+1)_n\,(1)_n}$$

$$=\frac{1}{[r]_![s]_!}\sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}\left(\frac{x}{2}\right)^{2m+r}\left(\frac{y\rho(x)}{2}\right)^{2n+s}}{\Gamma m+1\;\;\Gamma n+1\;\;\Gamma r+m+1\;\;\Gamma s+n+1}$$

$$=\sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}}{!\,n\;\;!\,m\;\;\Gamma r+m+1\;\;\Gamma s+n+1}\left(\frac{x}{2}\right)^{2m+r}\left(\frac{y\,\rho(x)}{2}\right)^{2n+s}$$

Hence, we get

(15)

$$\lim_{q\to1} J_{r,s}\big((1-q)x\,,(1-q)y\,;q\big) = J_{r,s}\,(x,y)$$

IV. RECURRENCE RELATION FOR $J_{r,s}(x\,,y\,;q)$

Lemma 4.1:-If r ,s be integer then $J_{r,s}(x\,,y\,;q)$ satisfies

(16)

$$J_{-r,s}(x\,,y\,;q)=(-1)^r J_{r,s}(x\,,y\,;q)$$

Proof: Using q-Bessel function (7)

$$J_{r,s}(x\,,y\,;q)=\sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}\left(\frac{x}{2}\right)^{2m+r}\left(\frac{y\rho(x)}{2}\right)^{2n+s}}{[m+r]_q!\;[m]_q!\,[n+s]_q!\;[n]_q!}$$

Here Substitute r → -r and we get

$$J_{-r,s}(x\,,y\,;q)=\sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}\left(\frac{x}{2}\right)^{2m-r}\left(\frac{y\rho(x)}{2}\right)^{2n+s}}{[m-r]_q!\;[m]_q!\,[n+s]_q!\;[n]_q!}$$

Replace m by r +k then we obtain

$$= \sum_{k,n=0}^{\infty} \frac{(-1)^{r+k+n}\left(\frac{x}{2}\right)^{2k+r}\left(\frac{y\rho(x)}{2}\right)^{2n+s}}{[k+r]_q!\ [k]_q!\,[n+s]_q!\ [n]_q!}$$

$$= (-1)^r \sum_{k,n=0}^{\infty} \frac{(-1)^{k+n}\left(\frac{x}{2}\right)^{2k+r}\left(\frac{y\rho(x)}{2}\right)^{2n+s}}{[k+r]_q!\ [k]_q!\,[n+s]_q!\ [n]_q!}$$

$$= (-1)^r J_{r,s}(x\,,y\,;q)$$

Which is the recurrence relation then we have the following lemma (16)

$$J_{-r,s}(x\,,y\,;q) = (-1)^r J_{r,s}(x\,,y\,;q)$$

Lemma 4.2:-If r, s be integer then $J_{r,s}(x\,,y\,;q)$ satisfies

(17)

$$J_{r,-s}(x\,,y\,;q) = (-1)^s J_{r,s}(x\,,y\,;q)$$

Proof: Using Definition of q –Bessel function (7) and Substitute s→ -s and we get

$$J_{r,-s}(x\,,y\,;q) = \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n}\left(\frac{x}{2}\right)^{2m+r}\left(\frac{y\rho(x)}{2}\right)^{2n-s}}{[m-r]_q!\ [m]_q!\,[n-s]_q!\ [n]_q!}$$

Replace n by s+ t then we get

$$= \sum_{t,n=0}^{\infty} \frac{(-1)^{m+s+t}\left(\frac{x}{2}\right)^{2k+r}\left(\frac{y\rho(x)}{2}\right)^{2s+t}}{[k+r]_q!\ [k]_q!\,[t]_q!\ [s+t]_q!}$$

$$= (-1)^s \sum_{t,n=0}^{\infty} \frac{(-1)^{m+t}\left(\frac{x}{2}\right)^{2k+r}\left(\frac{y\rho(x)}{2}\right)^{2s+t}}{[k+r]_q!\ [k]_q!\,[t]_q!\ [s+t]_q!}$$

$$J_{r,-s}(x\,,y\,;q) = (-1)^s J_{r,s}(x\,,y\,;q)$$

Which is the prove of recurrence relation (17).

Lemma 4.3:-If r ,s be integer then $J_{r,s}(x\,,y\,;q)$ satisfies

(18)

$$J_{-r,-s}(x\,,y\,;q) = (-1)^r J_{r,-s}(x\,,y\,;q) = (-1)^s J_{r,-s}(x\,,y\,;q) = (-1)^{r+s} J_{r,s}(x\,,y\,;q)$$

Proof: Use (7) Substitute m→ r+k & n→ s+t

Then Result is Obtain

$$J_{-r,-s}(x\,,y\,;q) = (-1)^r J_{r,-s}(x\,,y\,;q) = (-1)^s J_{r,-s}(x\,,y\,;q) = (-1)^{r+s} J_{r,s}(x\,,y\,;q)$$

Lemma 4.4:- The function $J_{r,s}(x\,,y\,;q)$satisfies the relation (19)
$J_{r,s}(-x\,,y\,;q) = (-1)^r\ J_{r,s}(x\,,y\,;q)$
and hence it is even (or odd) function if the integer n is even (or odd).

Proof: We have by (7)

$$J_{r,s}\,(x,y\,;q) = \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n}\left(\frac{x}{2}\right)^{2m+r}\left(\frac{y\rho(x)}{2}\right)^{2n+s}}{[m+r]_q!\ [m]_q!\,[n+s]_q!\ [n]_q!}$$

for x=-x, then we get

$$J_{r,s}\,(-x,y\,;q) = \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n}\left(\frac{-x}{2}\right)^{2m+r}\left(\frac{y\rho(x)}{2}\right)^{2n+s}}{[m+r]_q!\ [m]_q!\,[n+s]_q!\ [n]_q!}$$

Then

$$J_{r,s}\,(x,y\,;q) = (-1)^{2m+r} \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n}\left(\frac{x}{2}\right)^{2m+r}\left(\frac{y\rho(x)}{2}\right)^{2n+s}}{[m+r]_q!\ [m]_q!\,[n+s]_q!\ [n]_q!}$$

For all value of m is positive$(-1)^{2m} = 1$
Then

$$J_{r,s}\,(x,y\,;q) = (-1)^r \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n}\left(\frac{x}{2}\right)^{2m+r}\left(\frac{y\rho(x)}{2}\right)^{2n+s}}{[m+r]_q!\ [m]_q!\,[n+s]_q!\ [n]_q!}$$

Hence the recurrence relation is

$$J_{r,s}(-x\,,y\,;q) = (-1)^r J_{r,s}(x\,,y\,;q)$$

Lemma 4.5:- The function $J_{r,s}(x\,,y\,;q)$satisfies the relation (20)

$$J_{r,s}(x\,,-y\,;q) = (-1)^s\ J_{r,s}(x\,,y\,;q)$$

and hence it is even (or odd) function if the integer n is even (or odd).
Now, if we substitute y by –y in the relation (7) , then we get the result.

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Lemma 4.6:- The function $J_{r,s}(x\,,y\,;q)$satisfies the relation
(21)

$$J_{r,s}(-x\,,-y\,;q) = (-1)^{r+s}\, J_{r,s}(x\,,y\,;q)$$

And hence it is even (or odd) function if the integer n is even (or odd).

Now, if we substitute x by –x (same as above) & y by –y in the relation (7) , then we get the result.

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